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Classical information deficit and monotonicity on local operations

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Abstract

We investigate the so-called *classical information deficit*—a candidate for a measure of classical correlations emerging from a thermodynamical approach initiated by Oppenheim *et al* (*Phys. Rev. Lett.* **89** 180402 (*Preprint* quant-ph/0112074)). It is defined as the difference between the amount of information that can be concentrated by the use of LOCC and the information contained in subsystems. We compare a one-way version of this quantity with a measure of classical correlations proposed by Henderson and Vedral. As a result, we obtain that the quantity can increase under local operations, hence it does not possess the property required for a good measure of classical correlations. Recently it was shown by Devetak 2004 (*Preprint* quant-ph/0406234) that the regularized version of this quantity is monotonic under local operations. In this context, our result implies that regularization plays a role of 'monotonizer'.

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Introduction

Correlations are a fundamental property of compound quantum distributed systems. The study of *quantum* correlations was initiated by Einstein, Podolsky and Rosen and Schrödinger. They were concerned with entanglement–quantum correlation, which are nonexistent in classical physics. Usefulness of entanglement in quantum information theory to such tasks as teleportation or dense coding has motivated its extensive study. However, subsequently an important subject of characterizing other interesting types of correlations has emerged. Namely, quantum correlation has been studied as a notion independent of entanglement [1, 3]. There have been trials to divide total correlation into classical and quantum parts [4, 5]. A measure of 'classical or quantum' correlations was defined and analysed in [6] and strange properties of classical correlation of quantum states were discovered in [7]. A measure of classical correlation has also been proposed in [8].

In [1] an operational measure of quantum correlations was proposed. It was based on the idea that by using a system in the state ϱ one can draw $(N - S(\varrho))kT \ln 2$ of work from a single heat bath, where N is the amount of qubits in state ϱ and $S(\varrho)$ is the von Neumann entropy of a given state. So the information function given by

$$I(\varrho) = N - S(\varrho) \tag{1}$$

can be treated as equivalent to work (see [9] in this context). This scenario was used in the distributed quantum system, where Alice and Bob are allowed to perform only local operations and communicate classically with each other (these are so-called LOCC operations) to concentrate information contained in the state on local subsystems. For nonclassical states the amount of work drawn by LOCC (or equivalently, the amount of information I_{LOCC} we can concentrate by LOCC operation on local subsystems) is usually smaller than the work extractable by global operations (or equivalently, the information I_{GO}, to which we have access by using global operation). The resulting difference $\Delta = I_{\mathrm{GO}} - I_{\mathrm{LOCC}}$ is called the information deficit or work deficit and it accounts for the part of correlation that must be lost during classical communication, and thus describes purely quantum correlation. One can consider a one-way version of the quantity, where only one-way classical communication is allowed. If communication is from Alice to Bob, the one-way deficit is denoted by $\Delta^{\rightarrow}$.

In [5] a complementary quantity that could account for classical correlation was defined—the classical information deficit Δ_{cl}:

$$\Delta_{\mathrm{cl}} = I_{\mathrm{LOCC}} - I_{\mathrm{LO}} \tag{2}$$

where I_{LO} is the amount of information accessible by using only local operations performed on N_A qubits of subsystem A and N_B qubits of subsystem B (i.e. $I_{\mathrm{LO}} = N_A - S(\varrho_A) + N_B - S(\varrho_B)$). One can see that the two measures of correlations add up to quantum mutual information given by

$$I_M = S(\varrho_A) + S(\varrho_B) - S(\varrho_{AB}) \tag{3}$$

where $S(\varrho) = -\mathrm{Tr}\,\varrho \log \varrho$ is the von Neumann entropy of the state ϱ and $\varrho_{A(B)} = \mathrm{Tr}_{B(A)}\varrho_{AB}$, i.e. we have

$$\Delta_{\mathrm{cl}} + \Delta = I_M. \tag{4}$$

Analogously, we have the following formula for the one-way version of the classical information deficit:

$$\Delta_{\mathrm{cl}}^{\rightarrow}(\varrho_{AB}) = I_{\mathrm{LOCC}}^{\rightarrow} - I_{\mathrm{LO}}. \tag{5}$$

In this paper we analyse connections between the classical one-way deficit and a measure of classical correlation C_{HV} introduced by Henderson and Vedral in [4]. The latter is maximal Bob's average information that can be obtained by Alice performing measurement on the system and telling Bob the outcomes. We provide a formula for $\Delta_{\mathrm{cl}}^{\rightarrow}$ and obtain that it is very close to the formula for the latter measure. We examine when the two measures can be equal to each other. On the basis of this we find that whenever $\Delta_{\mathrm{cl}}^{\rightarrow}$ and C_{HV} are distinct, $\Delta_{\mathrm{cl}}^{\rightarrow}$ can increase under local operations. We provide explicit examples of states for which C_{HV} and $\Delta_{\mathrm{cl}}^{\rightarrow}$ are indeed distinct, showing thereby a nonintuitive fact, i.e. that the one-way classical information deficit $\Delta_{\mathrm{cl}}^{\rightarrow}(\varrho_{AB})$ is not a measure of classical correlation, because it can *increase* under local operations. Remarkably, it was recently shown by Devetak [2] that the quantity which is equal to the regularized classical information deficit $\Delta_{\mathrm{cl}}^{\rightarrow\infty}(\varrho_{AB})$ is monotonic under LO. Combining those results with ours, we obtain that regularization may play a role of monotonizer. An asymptotic version of a function may be monotonic, even though one copy version is not.

One of the methods that we applied to obtain our results was to find a bridge between the results from the field of classical capacity of quantum channels [10–13] and our quantities. Then some peculiar results obtained in the latter field allowed us to find states suitable for our problems.

Formula for $\Delta_{\mathrm{cl}}^{\rightarrow}$ and comparison with Henderson–Vedral measure

In this section we provide a formula for $\Delta_{\mathrm{cl}}^{\rightarrow}$ and compare it with a measure of classical correlation introduced by Henderson and Vedral. To this end we have to determine a formula for the maximal amount of information which can be concentrated on subsystems via a protocol, in which one-way classical communication is allowed. The most general such protocol is the following. Alice makes a measurement on her part of state and tells her results to Bob. The amount of concentrable information is then equal to the information of Alice plus average final information of Bob. The protocol transforms the state in the following way:

$$\varrho_{AB} \to \varrho'_{AB} = \sum_i P_i \otimes I \varrho_{AB} P_i \otimes I \tag{6}$$

where p_i given by

$$p_i = \mathrm{Tr}(P_i \otimes I \varrho_{AB} P_i \otimes I) \tag{7}$$

is the probability that Bob gets the state ϱ_i^B, which is of the form

$$\varrho_i^B = \mathrm{Tr}_A(P_i \otimes I \varrho_{AB} P_i \otimes I)/p_i \tag{8}$$

and $\{P_i\}$ are projectors constituting von Neumann measurement. Usually, in LOCC paradigm one would allow for POVM. However, POVM requires adding ancillas, which we have to take into account if we are estimating the amount of information that we can concentrate. Thus, we include from the very beginning all needed ancillas and consider von Neumann measurement. In such a way we take into account POVMs, too. (There is an open question of whether it pays to add ancillas at all; we will discuss this later.)

The amount of information $I(\mathcal{P})$ which can be concentrated on subsystems in the one-way protocol $\mathcal{P}$ is thus equal to

$$I(\mathcal{P}) = I_A^{\mathrm{out}} + \overline{I_B^{\mathrm{out}}} \tag{9}$$

$$= N_A - S(\varrho'_A) + N_B - \sum_i p_i S(\varrho_B^i) \tag{10}$$

$$= N - S(\varrho'_A) - \sum_i p_i S(\varrho_B^i) \tag{11}$$

where $N_{A(B)}$ is the amount of qubits in the part of state of Alice (Bob), $(N_A + N_B = N)$, $\varrho'_A = \mathrm{Tr}_B \varrho'_{AB}$. The maximal information that can be concentrated by one-way protocols $\mathcal{P}^{\rightarrow}$ is denoted by $I^{\rightarrow}$:

$$I^{\rightarrow} = \sup_{\mathcal{P}^{\rightarrow}} I(\mathcal{P}^{\rightarrow}). \tag{12}$$

Having the formula for $I^{\rightarrow}$ we can express $\Delta_q^{\rightarrow}$ as

$$\Delta_q^{\rightarrow} = N - S(\varrho_{AB}) - \sup_{\{P_i\}} \left\{ N - \sum_i p_i S(\varrho_B^i) - S(\varrho'_A) \right\}$$

$$= \inf_{\{P_i\}} \left\{ \sum_i p_i S(\varrho_B^i) + S(\varrho'_A) \right\} - S(\varrho_{AB}). \tag{13}$$

Since $\Delta_{\rm cl}^{\rightarrow}$ is equal to the difference between total information $N - S(\varrho_{AB})$ and $I^{\rightarrow}$, we obtain

$$\Delta_{\rm cl}^{\rightarrow}(\varrho_{AB}) = \sup_{\{P_i\}}\left[\{S(\varrho_A) - S(\varrho_A')\} + \left\{S(\varrho_B) - \sum_i p_i S\big(\varrho_B^i\big)\right\}\right] \tag{14}$$

where the supremum is taken over all local dephasings on Alice's side. Note that the protocol is determined by choosing Alice's measurement. Note also that the optimal measurement is a complete one, i.e. P_i can be chosen to be one-dimensional projectors. Indeed, given any incomplete measurement, Alice can always refine it in such a way that her entropy will not increase, and of course, any refinement will not increase Bob's average entropy. In equation (14), we have distinguished two terms. The second term

$$S(\varrho_B) - \sum_i p_i S\big(\varrho_B^i\big) \tag{15}$$

shows the decrease of Bob's entropy after Alice's measurement. The first one

$$S(\varrho_A) - S(\varrho_A') \tag{16}$$

denotes the cost of this process on Alice's side, and is non-positive. It vanishes only if Alice measures in the eigenbasis of her local density matrix (ϱ_A). Thus, the expression for $\big(\Delta_{\rm cl}^{\rightarrow}\big)$ is very similar to the measure of classical correlation introduced by Henderson and Vedral [4]:

$$C_{\rm HV}(\varrho_{AB}) = \sup_{P_i}\left(S(\varrho_B) - \sum_i p_i S\big(\varrho_B^i\big)\right). \tag{17}$$

Originally, in the definition of $C_{\rm HV}$ the supremum was taken over POVMs, but as mentioned, we take the state acting already on a suitably larger Hilbert space, unless stated otherwise explicitly. The difference between the Henderson–Vedral classical correlation measure and that given in equation (14) is that the former does not include Alice's entropic cost of performing dephasing. Hence in general

$$\Delta_{\rm cl}^{\rightarrow} \leqslant C_{\rm HV}.$$

When $\Delta_{\rm cl}^{\rightarrow}$ can be equal to $C_{\rm HV}$

In this section we show that our two quantities can be equal if and only if there is a measurement that is optimal for both quantities and the measurement is in the eigenbasis of the density matrix of Alice's system. More precisely we prove the following lemma.

Lemma 1. *Let ϱ_{AB} be any bipartite state. Then $C_{\rm HV}(\varrho_{AB}) = \Delta_{\rm cl}^{\rightarrow}(\varrho_{AB})$ if and only if there exist projectors $\{P_i\}$ such that they commute with $\varrho_A(={\rm tr}_B\varrho_{AB})$ and they are optimal for both $C_{\rm HV}$ and $\Delta_{\rm cl}^{\rightarrow}$ for the state ϱ_{AB}.*

Remark 1. Note that eigenbasis of ϱ_A may not be unique.

Proof. For specific measurement, let us use the following notation:

$$c_{\rm HV} = S(\varrho_B) - \sum_i p_i S\big(\varrho_i^B\big) \tag{18}$$

$$\delta_{\rm cl}^{\rightarrow} = S(\varrho_A) - S(\varrho_A') + S(\varrho_B) - \sum_i p_i S\big(\varrho_i^B\big). \tag{19}$$

The quantities c_{HV} and $\delta_{\mathrm{cl}}^{\rightarrow}$ are functions of state and a measurement respectively. We have

$$C_{\mathrm{HV}} = \sup_{\{P_i\}} c_{\mathrm{HV}} \tag{20}$$

$$\Delta_{\mathrm{cl}}^{\rightarrow} = \sup_{\{P_i\}} \delta_{\mathrm{cl}}^{\rightarrow}. \tag{21}$$

'$\Rightarrow$' Let us prove the 'only if' part: suppose that

$$C_{\mathrm{HV}} = \Delta_{\mathrm{cl}}^{\rightarrow}.$$

Consider measurement (i) which achieves C_{HV} and measurement (ii) which achieves $\Delta_{\mathrm{cl}}^{\rightarrow}$. Let $c_{\mathrm{HV}}^{\mathrm{i(ii)}}$ be the values c_{HV} for measurement i(ii) and $S\big(\varrho'^{\mathrm{(ii)}}_A\big)$ is Alice's part entropy after measurement (ii). Then

$$\Delta_{\mathrm{cl}}^{\rightarrow} = S(\varrho_A) - S\big(\varrho'^{\mathrm{(ii)}}_A\big) + c_{\mathrm{HV}}^{\mathrm{(ii)}} = c_{\mathrm{HV}}^{\mathrm{(i)}} = C_{\mathrm{HV}}. \tag{22}$$

We know that for an arbitrary measurement $S(\varrho_A) - S(\varrho'_A) \leqslant 0$ [14] and $c_{\mathrm{HV}}^{\mathrm{(ii)}} \leqslant c_{\mathrm{HV}}^{\mathrm{(i)}}$. If we want equality (22) holds, then it must be that

$$S(\varrho_A) - S\big(\varrho_A^{\mathrm{(ii)}}\big) = 0 \qquad \text{and} \qquad c_{\mathrm{HV}}^{\mathrm{(ii)}} = c_{\mathrm{HV}}^{\mathrm{(i)}}. \tag{23}$$

It follows that measurement (ii) is also optimal for C_{HV}. Moreover, note that this measurement is made in eigenbasis, otherwise it would increase entropy $S(\varrho'_A)$ violating equation (22).

'$\Leftarrow$' The 'if' proof is obvious. Since we assume that the measurement achieving C_{HV} and $\Delta_{\mathrm{cl}}^{\rightarrow}$ is the same and is made in the eigenbasis of ϱ_A, so then $S(\varrho_A) - S(\varrho'_A) = 0$, so that $\Delta_{\mathrm{cl}}^{\rightarrow}$ and C_{HV} must be equal. This ends the proof of the lemma. □

When $\Delta_{\mathrm{cl}}^{\rightarrow}$ can increase under local operations

The main result of this section is to show that $\Delta_{\mathrm{cl}}^{\rightarrow}$ can increase under local operations. To this end we first show that whenever $\Delta_{\mathrm{cl}}^{\rightarrow}$ and C_{CV} are distinct, the former can increase under local operations. Next we provide examples of states for which it is the case.

Lemma 2. *If $\Delta_{\mathrm{cl}}^{\rightarrow} \neq C_{\mathrm{HV}}$ then the quantity $\Delta_{\mathrm{cl}}^{\rightarrow}$ can be increased by local operations.*

Therefore let us assume that $\Delta_{\mathrm{cl}}^{\rightarrow} < C_{\mathrm{HV}}$ for the state ϱ_{AB}. (Recall that, in general, $\Delta_{\mathrm{cl}}^{\rightarrow} \leqslant C_{\mathrm{HV}}$.) Let us consider an optimal measurement $\{P_i^{\mathrm{HV}}\}$ achieving C_{HV}. After the measurement, the state is of the form

$$\varrho'_{AB} = \sum_i p_i P_i^{\mathrm{HV}} \otimes \varrho_i^B.$$

We know that C_{HV} cannot increase after local operations [4]. Then

$$C_{\mathrm{HV}}(\varrho_{AB}) \leqslant C_{\mathrm{HV}}(\varrho'_{AB}) = S(\varrho_B) - \sum_i p_i S\big(\varrho_i^B\big) \tag{24}$$

so that $\{P_i^{\mathrm{HV}}\}$ is an optimal measurement for the state ϱ'_{AB} also. Now if we repeat the measurement $\{P_i^{\mathrm{HV}}\}$ on ϱ'_{AB} we get the same value of $C_{\mathrm{HV}}(\varrho'_{AB})$ as before since ϱ'_{AB} and the created Bob ensemble do not change under that particular measurement. Thus

$$C_{\mathrm{HV}}(\varrho_{AB}) = C_{\mathrm{HV}}(\varrho'_{AB}).$$

Note that $\{P_i^{\mathrm{HV}}\}$ corresponds to the eigenbasis of ϱ'_A, where ϱ'_A is the reduced matrix of ϱ'_{AB}. Then

$$\Delta_{\mathrm{cl}}^{\rightarrow}(\varrho'_{AB}) = C_{\mathrm{HV}}(\varrho'_{AB})$$

so that

$$\Delta_{\mathrm{cl}}^{\rightarrow}(\varrho'_{AB}) > \Delta_{\mathrm{cl}}^{\rightarrow}(\varrho_{AB})$$

i.e. $\Delta_{\mathrm{cl}}^{\rightarrow}$ is increased after local operations of dephasing by P_i.

Having proved lemma 2 the question is whether there exist states for which $\Delta_{\rm cl}^{\rightarrow} \neq C_{\rm HV}$. We know that in such cases, there should not exist any measurement optimizing both $\delta_{\rm cl}^{\rightarrow}$ and $c_{\rm HV}$, which is made in eigenbasis. Equivalently, there should not exist a measurement that optimizes $C_{\rm HV}$, which is made in the eigenbasis of ϱ_A. To show this, the following results of Schumacher and Westmoreland [15] and King, Nathanson and Ruskai [13] connected with classical capacity of a quantum channel are helpful.

Suppose a source produces states ϱ_k with probabilities p_k. For this ensemble, the authors in [13, 15] considered a quantity called entropy defect or Holevo quantity, defined as

$$\chi = S(\varrho) - \sum_k p_k S(\varrho_k)$$

where

$$\varrho = \sum_k p_k \varrho_k.$$

They were interested in maximizing χ for the output ensemble $\{p_k, \Lambda(|\psi_k\rangle\langle\psi_k|)\}$, where Λ is a fixed completely positive map (channel).

It turns out that for some channels, to maximize χ, one needs a non-orthogonal input ensemble. This was first shown by Fuchs [12]. An example of such a channel is given by the following map [15]:

$$\Lambda_1(\varrho) = A_1 \varrho A_1^\dagger + A_2 \varrho A_2^\dagger \tag{25}$$

where

$$A_1 = \sqrt{\tfrac{1}{2}}|1\rangle\langle 1| + |0\rangle\langle 0| \qquad A_2 = \sqrt{\tfrac{1}{2}}|0\rangle\langle 1|$$

where $\{|0\rangle, |1\rangle\}$ is the standard basis in $\mathbb{C}^2$. For this channel, maximum χ is obtained for non-orthogonal input states.

On the other hand, it has been recently shown [13] that sometimes the number of states in the optimal ensemble must be greater than the dimension of the system. An example is the map given by

$$\Lambda_2(\varrho) = \tfrac{1}{2}(I + [0.6w_1, 0.6w_2, 0.5 + 0.5w_3] \cdot \vec{\sigma}) \tag{26}$$

where

$$\varrho = \tfrac{1}{2}(I + \vec{w}\vec{\sigma})$$

and $\vec{w} = (w_1, w_2, w_3)$ with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and σ_i being the Pauli matrices. In this case χ is maximized by a three-component ensemble.

The above examples can lead us to a bipartite state ϱ_{AB}, for which $C_{\rm HV}$ is not achieved by the measurement in the eigenbasis of ϱ_A. (Note that these examples act only as indications. The results of channel capacities are not used directly, although such a direct connection is not ruled out.)

More precisely, given any channel and ensemble, we will construct some bipartite state and a measurement on one of its subsystems. We will expect that the measurement will give a high value of $c_{\rm HV}$ on that state. In particular, if the ensemble has two components but is non-orthogonal, we obtain a higher value of $c_{\rm HV}$ than the value produced by measurement in eigenbasis of ϱ_A, so that the latter measurement is no longer optimal. Moreover, if the ensemble has three components and the measurement gives a better value than any von Neumann measurement, the optimal measurement for attaining $C_{\rm HV}$ will not be a von Neumann measurement, but POVM. We show that it is indeed the case in both situations.

Let us now present our construction of the state and measurement from a given channel Λ and ensemble $\{p_i, \psi_i\}$. We will first exhibit two ways of obtaining ensemble $\{p_i, \varrho_i\}$ from a pure bipartite state ψ_{AB}, where $\varrho_i = \Lambda(\psi_i)$. Let ψ_{AB} be a state for which

$$\mathrm{Tr}_A|\psi_{AB}\rangle\langle\psi_{AB}| = \sum_i p_i|\psi_i\rangle\langle\psi_i|.$$

One can write it in the form $|\psi_{AB}\rangle = \sum_i \sqrt{p_i}|i\rangle|\psi_i\rangle$, where $|i\rangle$ are orthogonal. Note that when we make a measurement in the basis $|i\rangle$ at Alice's side, the ensemble $\{p_i, \psi_i\}$ is created at Bob's side. Then one obtains ensemble $\{p_i, \varrho_i\}$ by letting ψ_i evolve through the channel Λ. But one can achieve $\{p_i, \varrho_i\}$ in a different way. First, the state ψ_{AB} is prepared and the operation $I_A \otimes \Lambda_B$ is performed, producing state ϱ_{AB}:

$$\varrho_{AB} = (I_A \otimes \Lambda_B)(\psi_{AB}).$$

Then Alice makes the measurement in the basis $|i\rangle$ and this produces the ensemble $\{p_i, \varrho_i\}$ at Bob's side. The connection between the scenarios is illustrated by the commuting diagram below. Starting from ψ_{AB}, we can achieve the ensemble $\{p_i, \varrho_i\}$ in two ways.

$$\begin{array}{ccc} \psi_{AB} & \xrightarrow{I_A\otimes\Lambda_B} & \varrho_{AB} \\ \Big| & & \Big| \\ M_A & & M_A \\ \downarrow & & \downarrow \\ \{p_i, \psi_i\}_B & \xrightarrow{\Lambda} & \{p_i, \varrho_i\}_B \end{array} \tag{27}$$

Here M_A denotes the measurement by Alice and $\{*, *\}_B$ denotes the corresponding ensemble at Bob's side. If we want to find the needed state ϱ_{AB} for which $\Delta_{\mathrm{cl}}^{\rightarrow} \neq C_{\mathrm{HV}}$, we should construct a pure state ψ_{AB} and then perform operation $I_A \otimes \Lambda_B$. First we use the channel (given by equation (25)) and ensemble from [15] to obtain ϱ_{AB} for which c_{HV} for some measurement is greater than that for the measurement in eigenbasis.

An example of a non-orthogonal ensemble, for the channel (Λ_1) given by equation (25), which gives greater χ than any orthogonal one is $\left\{\left\{\frac{1}{2}, \psi_0\right\}, \left\{\frac{1}{2}, \psi_1\right\}\right\}$, where

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{28}$$

$$|\psi_1\rangle = \frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle. \tag{29}$$

Then we have

$$\varrho_{AB} = \left(I^A \otimes \Lambda_1^B\right)|\psi_{AB}\rangle\langle\psi_{AB}| \tag{30}$$

where

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A\,|\psi_0\rangle_B + |1\rangle_A\,|\psi_1\rangle_B).$$

Now, we can check directly that for Alice's measurement in the basis $|0\rangle, |1\rangle$ (which prepares the non-orthogonal ensemble $\left\{\left\{\frac{1}{2}, \psi_0\right\}, \left\{\frac{1}{2}, \psi_1\right\}\right\}$ on Bob's side), the Henderson–Vedral quantity c_{HV} (see equation (18)) attains the value $c_{\mathrm{HV}}^{(1)} = 0.456\,67$. But for Alice's measurement in the eigenbasis of $\varrho_A, c_{\mathrm{HV}}$ attains the value $c_{\mathrm{HV}}^{(2)} = 0.3356$. Therefore

$c_{\rm HV}^{(1)} > c_{\rm HV}^{(2)}$, i.e. there exists Alice's measurement which gives better value for $c_{\rm HV}$ than the measurement in the eigenbasis of ϱ_A. The optimal measurement is therefore clearly not in eigenbasis. This fact, as follows from lemma 1, implies that for a state given by formula (30) one has $C_{\rm HV} \neq \Delta_{\rm cl}^{\rightarrow}$ and, even more remarkably, as follows from lemma 2, for this state $\Delta_{\rm cl}^{\rightarrow}$ increases under local measurements. The operation that increases $\Delta_{\rm cl}^{\rightarrow}$ is Alice's dephasing in the basis $|0\rangle, |1\rangle$.

We now use the results of [13] to find the next example of state for which $C_{\rm HV} \neq \Delta_{\rm cl}^{\rightarrow}$. The three-component ensemble for which χ, for the channel Λ_2 given by equation (26), is greater than that for any two-component ensemble is $\{p_i, |\phi_i\rangle\}$ where $p_0 = 0.4023$, $p_1 = p_2 = 0.298\,85$ and

$$\begin{aligned} |\phi_0\rangle &= |0\rangle \\ |\phi_1\rangle &= a|0\rangle + b|1\rangle \\ |\phi_2\rangle &= a|0\rangle - b|1\rangle \end{aligned}$$

with $a = 0.070\,1579$, $b = 0.821\,535$. Then by our prescription, the state for which POVM is better than any von Neumann measurement, as far as $C_{\rm HV}$ is concerned, is

$$\varrho_{AB} = \left(I^A \otimes \Lambda_2^B\right)|\phi_{AB}\rangle\langle\phi_{AB}|$$

where

$$|\phi_{AB}\rangle = \sum_{i=0}^{2} \sqrt{p_i}\,|i\rangle|\phi_i\rangle.$$

Again, as from lemmas 1 and 2, for this state $\Delta_{\rm cl}^{\rightarrow} \neq C_{\rm HV}$, hence $\Delta_{\rm cl}^{\rightarrow}$ can be increased by Alice's dephasing in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$, which can be treated as a POVM, since Alice's subsystem has rank 2, so it is efficiently qubit. For the measurement in the basis $|i\rangle$ (when the ensemble $\{p_i, |\phi_i\rangle\}$ is prepared on Bob's side), $c_{\rm HV}$ attains the value $\delta_{\rm HV}^1 = 0.324\,99$. For von Neumann measurements, $c_{\rm HV} \leqslant 0.321\,915$. Equality is obtained for the measurement in eigenbasis.

Finally, let us show that POVMs that are good for $C_{\rm HV}$ can be very bad for $\Delta_{\rm cl}$. One can check that the same POVM which gives high value for $c_{\rm HV}$ gives $\delta_{\rm cl}^{\rightarrow} < 0$. Therefore a POVM which is good for $c_{\rm HV}$ can be very bad for $\delta_{\rm cl}^{\rightarrow}$.

We have checked that for $\delta_{\rm cl}^{\rightarrow}$, the best von Neumann measurement is in eigenbasis. Then $\delta_{\rm cl}^{\rightarrow}$ attains the value $\delta_{\rm cl}^{\rightarrow(vN)} \approx 0.321\,915$.

This example indicates that $\Delta_{\rm cl}^{\rightarrow}$ might be such a quantity for which POVMs are not helpful. We conjecture that it can be a truth.

Asymptotic regime: regaining monotonicity

In this section we will describe briefly the Devetak result [2] on the asymptotic equality of classical deficit and the Henderson–Vedral quantity and show how it is related to ours.

To start with, one can consider an asymptotic version of classical deficit, which takes into account possible gain when one is operating on many copies collectively. It is given by

$$\Delta_{\rm cl}^{\rightarrow\infty} = \lim_n \frac{\Delta_{\rm cl}^{\rightarrow}(\rho^{\otimes n})}{n}. \tag{31}$$

Such operation is called 'regularization'. The regularized $\Delta_{\rm cl}^{\rightarrow}$ has intepretation of maximal local information that can be obtained per input copy from many copies of the given state by closed local operations and one-way communication. Equivalently, it is the maximal number of asymptotically pure qubits that can be obtained in such a way. Similarly, one can consider regularized C_{HV} quantity

$$C_{\mathrm{HV}}^{\infty} = \lim_{n} \frac{C_{\mathrm{HV}}(\rho^{\otimes n})}{n}. \tag{32}$$

This quantity was shown in [8] to have operational meaning of maximal amount of common random bits obtained by one-way LOCC operations in excess of communication invested.

As we have shown, in the one copy case $\Delta_{\mathrm{cl}}^{\rightarrow}$ is smaller than C_{HV} because Alice's optimal measurement produced additional entropy. Now, there is a question: can this production of entropy be overcome when Alice can perform joint measurements on many copies?

In [2] it was shown that this is indeed the case. Namely, for any Alice's POVM $\mathcal{P}$ on a single copy of ρ_{AB}, there exists a Alice's POVM $\mathcal{P}'$ on $\rho_{AB}^{\otimes n}$, such that after the measurement, Alice's entropy is $n(S(\rho_A) + \delta)$ while Bob's information gain, when he gets to know the outcomes of $\mathcal{P}'$, amounts to $n(I_B - \epsilon)$, where I_B is Bob's average information gain given outcomes of $\mathcal{P}$. Both ϵ and δ can be made arbitrarily small when n is sufficiently large. Thus the POVM $\mathcal{P}'$ provides asymptotically the same Bob's information gain per copy as the POVM $\mathcal{P}$, while increasing Alice's entropy only by a negligible amount. Consequently, from formula (14) it follows that the difference between $\Delta_{\mathrm{cl}}^{\rightarrow}(\rho^{\otimes n})$ and $C_{\mathrm{HV}}(\rho^{\otimes n})$ can be made equal to $n(\epsilon + \delta)$. If we take high n, so that ϵ and δ are small, and use it in the regularization formulae (31) and (32), one obtains that $C_{\mathrm{HV}}^{\infty} = \Delta_{\mathrm{cl}}^{\rightarrow\infty}$.

Now we can go back to the monotonicity question. It is easy to see that regularization does not affect monotonicity under local operations. Therefore C_{HV}^{∞} is still monotonic. Consequently, $\Delta_{\mathrm{cl}}^{\rightarrow\infty}$ is monotonic too. Thus classical deficit is a quantity which in the single copy case is not monotonic under local operations; however, after regularization it acquires monotonicity.

Discussion

In this paper we have considered classical information deficit $\Delta_{\mathrm{cl}}^{\rightarrow}$ defined as the difference between the amount of information that can be concentrated by LOCC and information concentrable by LO. It is equal to the difference between the measure of total correlation and the measure of quantum correlation present in the state. It was reasonable to expect that it should be a measure of classical correlation. We have shown that it is not true, because $\Delta_{\mathrm{cl}}^{\rightarrow}$ can increase under local operations. We have proved it through comparison with a measure of classical correlation proposed by Henderson–Vedral. We based this on a lemma which tells us when these quantities can be equal. We showed that if they are different, then $\Delta_{\mathrm{cl}}^{\rightarrow}$ can increase under local actions. The last thing we did was found examples of states for which $\Delta_{\mathrm{cl}}^{\rightarrow} \neq C_{\mathrm{HV}}$. We also exhibited an example, where POVM is very good for C_{HV}, but completely bad for $\Delta_{\mathrm{cl}}^{\rightarrow}$. This suggests that POVMs may not be helpful in the one-way protocol of localizing information. This would be compatible with the result for two-way protocols, where borrowing ancillas does not help in concentrating information [16]. The above results would mean that $\Delta_{\mathrm{cl}}^{\rightarrow}$ is useless as far as classical correlation of quantum states is concerned. Fortunately, it is not the case.

Recently it was shown in [2] that the regularized version of $\Delta_{\mathrm{cl}}^{\rightarrow}$ is a measure of classical correlation, because it is equal to distillable common randomness [8], which in fact is equal to the regularized C_{HV}. Since the latter is monotonic under local operation, then $\Delta_{\mathrm{cl}}^{\rightarrow}$ if regularized is also monotonic. It is a very puzzling fact that we have a quantity which defined that one copy of state can increase after local operations, but its regularized version cannot. Thus, according to our results, the regularization plays a role of 'monotonizer' in this case.

Moreover, it follows that $\Delta_{\mathrm{cl}}^{\rightarrow}$ is not additive. Indeed, we have found examples where $\Delta_{\mathrm{cl}}^{\rightarrow}$ and C_{HV} are not equal to each other, while they become equal after regularization by [2]. Since both of them can only go up under regularization, we obtain that $\Delta_{\mathrm{cl}}^{\rightarrow}$ of n copies is greater than n times $\Delta_{\mathrm{cl}}^{\rightarrow}$ of one copy for some states.

Finally, we hope that the results presented will stimulate further research towards understanding the quantities emerging from the thermodynamical approach to quantum distributed systems in the context of correlations contained in those systems.

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